

Final

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.

Problem 1 :

Prove that if $\phi : G \rightarrow G'$ is a group isomorphism, then the inverse function ϕ^{-1} is also a group isomorphism.

Solution :

ϕ^{-1} is clearly a bijective map, since its inverse is $(\phi^{-1})^{-1} = \phi$, so we only need to show that ϕ^{-1} is a group homomorphism. Let then $\alpha, \beta \in G'$, and consider their pre-images $a = \phi^{-1}(\alpha)$, $b = \phi^{-1}(\beta) \in G$. Since ϕ is a group homomorphism we have

$$\phi(\phi^{-1}(\alpha) \cdot \phi^{-1}(\beta)) = \phi(a \cdot b) = \phi(a)\phi(b) = \alpha \cdot \beta.$$

If we then apply the function ϕ^{-1} to left and right hand sides of the above equation we get $\phi^{-1}(\alpha) \cdot \phi^{-1}(\beta) = \phi^{-1}(\alpha \cdot \beta)$, thus proving that ϕ^{-1} is a group homomorphism.

Problem 2 :

Let D_n be the dihedral group of order $2n$.

1. Give the properties that the generators of a Dihedral group must satisfy which are enough to define it ;
2. Express $g = yx^{n-2}yx^5$ in the form $y^i x^j$ with $i, j \geq 0$.

Solution :

By definition of the group of order D_n , we have $y^{-1} = y$, $yx = x^{n-1}y = x^{-1}y$, and also

$$yx^{-1} = x(x^{-1}y)x^{-1} = x(yx)x^{-1} = xy;$$

then by an easy induction we get $yx^k = x^{-k}y$ for all $k \in \mathbb{Z}$. Hence :

$$g = (yx^{n-2})(y)x^5 = (x^{2-n}y)(y^{-1})x^5 = x^2(yy^{-1})x^5 = x^7 = y^0x^7$$

Problem 3 :

Let H and N be subgroups of a group G , and assume that N is normal. We denote their product

$$HN = \{hn|h \in H, n \in N\} \subset G$$

and similarly for NH . Prove that $HN = NH$ and this set is the smallest subgroup of G containing both H and N .

Solution :

We have $hn = (hnh^{-1})h \in NH$ since N is normal, so that $HN \subset NH$. Similarly $nh = h(h^{-1}nh) \in HN$, again since N is normal, so that $NH \subset HN$. Hence $HN = NH$, as we wanted. We will denote this set by X . To prove that X is a subgroup, given $a, b \in X$, we need to show that $a^{-1} \in X$ and $ab \in X$. We can write $a = hn \in HN = X$ and $b = h'n' \in HN = X$. Hence $a^{-1} = n^{-1}h^{-1} \in NH = X$ and $ab = hnh'n' = (hh')((h')^{-1}nh')n' \in HN = X$. Finally, obviously $H = \{h1|h \in H\} \subset HN = X$ and $N = \{1n|n \in N\} \subset HN = X$, and if $K \subset G$ is a subgroup containing both H and N , then it must contain $HN = X$. Therefore X is the smallest subgroup of G containing H and N .

Problem 4 :

Let G be the multiplicative group of invertible upper triangular matrices.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$, $a, c \neq 0$.

Let $S \subset G$ be the subset of all the matrices with $c = 1$. Prove that S is a normal subgroup, by showing that S is the kernel of a suitable group homomorphism.

Solution :

The group structure in G is given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta + b\gamma \\ 0 & c\gamma \end{pmatrix}$$

It immediately follows from the above identity that the map $G \rightarrow \mathbb{R} \setminus \{0\}$ is given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto c$$

is a group homomorphism from G to the multiplicative group $\mathbb{R} \setminus \{0\}$, and its kernel is exactly the subset S . Therefore $S \subset G$ must be a normal subgroup.

Problem 5 :

Show that every $\sigma \in S_n$ is a product of the n -cycle $\alpha = (1, 2, \dots, n)$ and the 2-cycle $\tau = (1, 2)$. (Hint : compute $\alpha^k \tau \alpha^{-k}$).

Solution :

Note that $\alpha^k \tau \alpha^{-k} = (k+1, k+2)$ for $k = 1, \dots, n-2$. Thus, we can generate transpositions of the form $(1, 2), (2, 3), \dots, (n-1, n)$.

Problem 6 :

1. Give the properties that the map

$$\begin{aligned} \alpha : G \times X &\rightarrow X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

satisfies so that it defines a group action.

2. Suppose that α defines a group action. For each $g \in G$, define the map $\tau_g : X \rightarrow X$ by $\tau_g(x) = \alpha(g, x)$, for all $x \in X$. Show that τ_g is a permutation of the set X .
3. Show that the map $\xi : G \rightarrow S_X$ defined by $\xi(g) = \tau_g$ is a group homomorphism where S_X denotes the group of all permutations of X .
4. Conversely, given a group homomorphism $\phi : G \rightarrow S_X$ from a group G to the group S_X of permutation of a set X , show that the map $\alpha : G \times X \rightarrow X$ defined by

$$\alpha(g, x) = \phi(g)(x)$$

is a group action.

Solution :

Class note.

Problem 7 :

1. Let G be a group and H, N be subgroups of G . Give two equivalent ways to define a internal semi direct product of N with H .
2. Let G be a group of order 6.
 - (a) Which theorem permits to insure the existence of a subgroup H_2 of order 2 and H_3 of order 3?

- (b) Why G is then a semi-direct product constructed with the group H_2 and H_3 ?
- (c) Describe all the possible semi-direct products (write all the details). Identify the dihedral group D_3 and the permutation group S_3 .
- (d) How many group of order 6 non isomorphic are there ? Explain your answer.

Solution :

Class note.

Problem 7 :

1. Prove that we have a canonical isomorphism

$$(S^1, \cdots) \simeq \left(\frac{\mathbb{R}}{\mathbb{Z}}, +\right)$$

where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

2. Prove that there is no cross section for $\frac{\mathbb{R}}{\mathbb{Z}}$.

Solution :

Class note.